

# ONE-DIMENSIONAL FORWARD-FORWARD MEAN-FIELD GAMES

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ABSTRACT. While the general theory for the terminal-initial value problem for mean-field games (MFGs) has achieved a substantial progress, the corresponding forward-forward problem is still poorly understood – even in the one-dimensional setting. Here, we consider one-dimensional forward-forward MFGs, study the existence of solutions and their long-time convergence. First, we discuss the relation between these models and systems of conservation laws. In particular, we identify new conserved quantities and study some qualitative properties of these systems. Next, we introduce a class of wave-like equations that are equivalent to forward-forward MFGs, and we derive a novel formulation as a system of conservation laws. For first-order logarithmic forward-forward MFG, we establish the existence of a global solution. Then, we consider a class of explicit solutions and show the existence of shocks. Finally, we examine parabolic forward-forward MFGs and establish the long-time convergence of the solutions.

## 1. INTRODUCTION

Mean-field games (MFGs) are models for large populations of competing rational agents who seek to optimize an individual objective function. A typical model is the *backward-forward MFG*. In one dimension, this game is determined by following the system of partial differential equations (PDEs):

$$\begin{cases} -u_t + H(u_x) = \varepsilon u_{xx} + g(m), \\ m_t - (H'(u_x)m)_x = \varepsilon m_{xx}. \end{cases} \quad (1.1)$$

For convenience, the spatial domain, corresponding to the variable  $x$ , is the 1-dimensional torus,  $\mathbb{T}$ , identified with the interval  $[0, 1]$ . The time domain, corresponding to the variable  $t$ , is the interval  $[0, T]$  for some terminal time,  $T > 0$ . The unknowns in the above system are  $u : \mathbb{T} \times [0, T] \rightarrow \mathbb{R}$  and  $m : \mathbb{T} \times [0, T] \rightarrow \mathbb{R}$ . In this game, each agent seeks to solve an optimal control problem. The function  $u(x, t)$  is the value function for this control problem for an agent located at  $x \in \mathbb{T}$  at the time  $t$ . This control problem is determined by a Hamiltonian,  $H : \mathbb{R} \rightarrow \mathbb{R}$ ,  $H \in C^2$ , and a coupling between each agent and the mean field,  $m$ , given by the function  $g : \mathbb{R}^+ \rightarrow \mathbb{R}$ ,  $g \in C^1$ . The first equation in (1.1) is a Hamilton-Jacobi equation and expresses the optimality of the value function,  $u$ . For each  $t \in [0, T]$ ,  $m$  is a probability density in  $\mathbb{T}$ . The second equation of (1.1), the Fokker-Planck equation, determines the evolution of  $m$ . The parameter  $\varepsilon \geq 0$  is the viscosity coefficient in the Fokker-Planck equation;  $\varepsilon = 0$  corresponds to *first-order MFGs* and  $\varepsilon > 0$  to *parabolic MFGs*. The system (1.1) is endowed with terminal-initial conditions; the

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*Date:* June 30, 2016.

*Key words and phrases.* Mean-field games; systems of conservation laws; convergence to equilibrium; Hamilton-Jacobi equations; transport equations; Fokker-Planck equations.

The authors were supported by KAUST baseline and start-up funds.

initial value of  $m$  is prescribed at  $t = 0$  and the terminal value of  $u$ , at  $t = T$ :

$$\begin{cases} u(x, T) = u_T(x) \\ m(x, 0) = m_0(x). \end{cases} \quad (1.2)$$

As a result, (1.1)-(1.2) is called the *terminal-initial value problem* or the *backward-forward MFG*.

Here, we examine a related model, the *forward-forward MFG* problem. This model is constructed by the reversal of the time variable in the Hamilton-Jacobi equation in (1.1). Accordingly, the *forward-forward MFG system* in  $\mathbb{T} \times [0, T]$  is determined by

$$\begin{cases} u_t + H(u_x) = \varepsilon u_{xx} + g(m) \\ m_t - (H'(u_x)m)_x = \varepsilon m_{xx}, \end{cases} \quad (1.3)$$

together with the *initial-initial condition*:

$$\begin{cases} u(x, 0) = u_0(x) \\ m(x, 0) = m_0(x). \end{cases} \quad (1.4)$$

The forward-forward model was introduced in [1] to approximate stationary MFGs. The key insight is that the parabolicity in (1.3) should imply the long-time convergence to a stationary solution. In the preceding MFG, a typical Hamiltonian,  $H$ , is the quadratic Hamiltonian,  $H(p) = \frac{p^2}{2}$ , or for  $\gamma > 1$ , the power-like Hamiltonian,  $H(p) = \frac{1}{\gamma}|p|^\gamma$  or  $H(p) = (1 + p^2)^{\frac{\gamma}{2}}$ . Regarding the coupling nonlinearity,  $g$ , here, we consider the power-like case,  $g(m) = m^\alpha$  for some  $\alpha > 0$ , or the logarithmic case,  $g(m) = \ln m$ .

Considerable research has focused on proving the existence of solutions for backward-forward MFGs. For example, weak solutions for parabolic problems were considered in [28, 31], strong solutions for parabolic problems in [23, 24, 28], and weak solutions for first-order MFGs in [9, 10]. The stationary case was also investigated in detail since it was first considered in [27]. For this case, the existence of classical and weak solutions was investigated in [18, 19, 21, 22]. The uniqueness of solution is well understood (both for stationary and time-dependent MFGs) via the monotonicity method introduced in [27, 28]. Monotonicity properties are also fundamental for the existence theory developed in [17]. One-dimensional MFGs provide examples and guidance for the study of higher-dimensional problems and numerical methods [3]. Moreover, these games have an independent interest in problems in networks and graphs [5, 7, 8] and congestion [19, 20].

In contrast to that of the backward-forward case, our understanding of forward-forward MFGs is limited. In particular, the existence and the long-time convergence of the forward-forward model have not been addressed, except in a few cases, see [25] and [29]. In [29], the forward-forward problem was examined in the context of eductive stability of stationary MFGs with a logarithmic coupling. In [25], the existence and regularity of solutions for the forward-forward, uniformly parabolic MFGs with subquadratic Hamiltonians was proven. Except for these cases, the question of existence and regularity is open in all other regimes. In the case of forward-forward MFGs without viscosity, these questions are particularly challenging. Moreover, the long-time convergence has not been established even in the parabolic case. Nevertheless, numerical results in [2] and [12] indicate that convergence holds and that the forward-forward model approximates well stationary solutions.

Not only as an effective tool to approximate stationary problems, the forward-forward MFGs can also be regarded as a learning game. In backward-forward MFGs, the density of the agents is transported by the (future) optimal trajectories of an

optimal control problem. In the forward-forward model, the interpretation the evolution of the agents is less straightforward. In this model, the density is transported by past optimal trajectories because the corresponding control problem has initial data, not terminal data. Thus, the actions of the agents are determined by a learning strategy where past densities drive their evolution.

This paper is structured as follows. In Section 2, we reformulate (1.1)-(1.2) and (1.3)-(1.4) as systems of conservation laws. There, we identify new conserved quantities for these problems in the case where  $\varepsilon = 0$ . Conserved quantities are fundamental in analyzing PDEs and in testing and validating numerical methods. Here, they are used in the long-time convergence analysis. Next, in Section 3, we derive wave-type equations that are equivalent to (1.3)-(1.4). For example, for the first-order, logarithmic forward-forward model, we obtain the PDE

$$u_{tt} = (1 + u_x^2)u_{xx}.$$

The preceding equation is equivalent to an elastodynamics problem. The corresponding elastodynamics equations have entropy solutions when the stress function is monotone. Thus, we obtain the existence of solutions for the original MFG. In addition, using results from [4], we identify a class of explicit solutions for the logarithmic MFGs. These explicit solutions provide an example where shocks arise in the forward-forward model. Finally, in Section 4, we examine forward-forward parabolic MFGs. Here, the entropies identified in Section 2 play an essential role in our analysis of the long-time behavior of solutions. Due to the parabolicity, these entropies are dissipated and force the long-time convergence of the solutions of (1.3)-(1.4).

## 2. SYSTEMS OF CONSERVATION LAWS AND FIRST-ORDER MFGS

Here, we consider deterministic MFGs; that is,  $\varepsilon = 0$ . In this case, (1.1) and (1.3) are equivalent to conservation laws, at least for smooth enough solutions. In this preliminary section, we examine these conservation laws and identify conserved quantities. In Section 4, we use these conserved quantities to establish the long-time convergence of the parabolic forward-forward MFG (1.3).

Before proceeding, we recall some well-known results on systems conservation laws in one dimension. We consider a conservation law of the form

$$U_t + (F(U))_x = 0, \quad (2.1)$$

where  $U : \mathbb{R} \times \mathbb{T} \rightarrow \mathbb{R}^2$  is the unknown and  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is the flux function. We say that  $(E, Q)$  is an entropy/entropy-flux pair if

$$(E(U))_t + (Q(U))_x = 0 \quad (2.2)$$

for any smooth solution of (2.1). We note that (2.2) implies that  $E(U)$  is a conserved quantity if the solution  $U$  of (2.1) is smooth; that is,

$$\frac{d}{dt} \int_{\mathbb{T}} E(U) dx = - \int_{\mathbb{T}} (Q(U))_x dx = 0. \quad (2.3)$$

**2.1. Backward-Forward MFG.** Now, we assume that (1.1) has a smooth enough solution, for example,  $u, m \in C^2(\mathbb{T} \times (0, \infty)) \cap C(\mathbb{T} \times [0, \infty))$ . We set  $v = u_x$  and differentiate the first equation in (1.1) with respect to  $x$ . Accordingly, we obtain the following system

$$\begin{cases} v_t + (g(m) - H(v))_x = 0, \\ m_t - (mH'(v))_x = 0. \end{cases} \quad (2.4)$$

To investigate the existence of an entropy for (2.4), we look for an entropy/entropy-flux  $(E, Q)$  satisfying (2.2) for  $U = (v, m)$ . By expanding (2.2), we get

$$\frac{\partial E}{\partial v}v_t + \frac{\partial E}{\partial m}m_t + \frac{\partial Q}{\partial v}v_x + \frac{\partial Q}{\partial m}m_x = 0. \quad (2.5)$$

In light of (2.4), (2.5) becomes

$$\frac{\partial E}{\partial v}H'(v)v_x - \frac{\partial E}{\partial v}g'(m)m_x + \frac{\partial E}{\partial m}H'(v)m_x + \frac{\partial E}{\partial m}mH''(v)v_x + \frac{\partial Q}{\partial v}v_x + \frac{\partial Q}{\partial m}m_x = 0. \quad (2.6)$$

Thus,

$$\frac{\partial Q}{\partial v} = -\frac{\partial E}{\partial v}H'(v) - \frac{\partial E}{\partial m}mH''(v) \quad \text{and} \quad \frac{\partial Q}{\partial m} = \frac{\partial E}{\partial v}g'(m) - \frac{\partial E}{\partial m}H'(v). \quad (2.7)$$

Consequently, we obtain the following PDE for  $E$

$$\frac{\partial}{\partial m} \left( -\frac{\partial E}{\partial v}H'(v) - \frac{\partial E}{\partial m}mH''(v) \right) = \frac{\partial}{\partial v} \left( \frac{\partial E}{\partial v}g'(m) - \frac{\partial E}{\partial m}H'(v) \right). \quad (2.8)$$

After elementary computations, the above equation becomes

$$\frac{1}{H''(v)} \frac{\partial^2 E}{\partial v^2} + \frac{1}{P''(m)} \frac{\partial^2 E}{\partial m^2} = 0, \quad (2.9)$$

where

$$P''(m) = \frac{g'(m)}{m}. \quad (2.10)$$

The preceding equation has the following trivial solutions:

$$E(v, m) = \alpha v + \beta m, \quad \alpha, \beta \in \mathbb{R}.$$

By inspection, we can verify that the following two expressions solve (2.9):

$$E(v, m) = mv \quad \text{and} \quad E(v, m) = H(v) - P(m).$$

Moreover, if  $g$  is increasing,  $P$  is a convex function whereas if  $g$  is decreasing,  $P$  is concave.

Using separation of variables and writing

$$E = \Phi(v)\Psi(m),$$

we derive the following conditions

$$\begin{cases} \frac{1}{H''(v)} \frac{\Phi''(v)}{\Phi(v)} = \lambda \\ \frac{1}{P''(m)} \frac{\Psi''(m)}{\Psi(m)} = -\lambda. \end{cases}$$

The conditions above take a simple form when  $g(m) = \frac{m^2}{2}$ , which corresponds to  $P(m) = \frac{m^2}{2}$ , and  $H(v) = \frac{v^2}{2}$ , namely

$$\begin{cases} \Phi''(v) = \lambda \Phi(v) \\ \Psi''(m) = -\lambda \Psi(m). \end{cases}$$

Thus, we have solutions of the form  $\Phi(v) = e^{\pm\sqrt{\lambda}v}$  and  $\Psi(m) = e^{\pm i\sqrt{\lambda}m}$ , which have exponential growth or oscillation depending upon the sign of  $\lambda$ . In addition to these conservation laws, there are also polynomial conservation laws. For illustration, some of these are shown in Table 1. In Table 2, we present some conservation laws for the anti-monotone backward-forward MFG with  $g(m) = -\frac{m^2}{2}$ . These laws are straightforward to compute as the determining equations for  $E$  are

$$\frac{\partial^2 E}{\partial m^2} + \frac{\partial^2 E}{\partial v^2} = 0$$

Degree	$E(v, m)$
3	$v^3 - 3m^2v$
3	$m^3 - 3mv^2$
4	$-6m^2v^2 + m^4 + v^4$
4	$mv^3 - m^3v$
5	$-10m^2v^3 + 5m^4v + v^5$
5	$-10m^3v^2 + 5mv^4 + m^5$
6	$15m^4v^2 - 15m^2v^4 - m^6 + v^6$
6	$m^5v - \frac{10}{3}m^3v^3 + mv^5$

TABLE 1. Conservation laws for the backward-forward MFG with  $H(v) = \frac{v^2}{2}$  and  $g(m) = \frac{m^2}{2}$  up to degree 6.

Degree	$E(v, m)$
3	$3m^2v + v^3$
3	$3mv^2 + m^3$
4	$6m^2v^2 + m^4 + v^4$
4	$m^3v + mv^3$
5	$10m^2v^3 + 5m^4v + v^5$
5	$10m^3v^2 + 5mv^4 + m^5$
6	$15m^4v^2 + 15m^2v^4 + m^6 + v^6$
6	$m^5v + \frac{10}{3}m^3v^3 + mv^5$

TABLE 2. Conservation laws for the backward-forward MFG with  $H(v) = \frac{v^2}{2}$  and  $g(m) = -\frac{m^2}{2}$  up to degree 6.

in the monotone case and

$$\frac{\partial^2 E}{\partial m^2} - \frac{\partial^2 E}{\partial v^2} = 0$$

in the anti-monotone case. In both cases, these equations have solutions that are homogeneous polynomials in  $m$  and  $v$ . In the monotone case, these conservation laws are the real and imaginary parts of  $(m + iv)^k$ . In the anti-monotone case, some of the conservation laws are coercive and, thus, control the  $L^p$  norms of  $v$  and  $m$  (at least for smooth solutions).

**2.2. Forward-forward MFG.** As previously, we assume that (1.3) has a solution,  $u, m \in C^2(\mathbb{T} \times (0, \infty)) \cap C(\mathbb{T} \times [0, \infty))$ , and we set  $v := u_x$ . We differentiate the first equation in (1.3) with respect to  $x$  and obtain the system:

$$\begin{cases} v_t - (g(m) - H(v))_x = 0, \\ m_t - (mH'(v))_x = 0. \end{cases} \quad (2.11)$$

We begin by examining the entropies for (2.11); that is, we look for  $(E, Q)$  satisfying (2.2) for  $U = (v, m)$ . We expand (2.2) to get

$$\frac{\partial E}{\partial v} v_t + \frac{\partial E}{\partial m} m_t + \frac{\partial Q}{\partial v} v_x + \frac{\partial Q}{\partial m} m_x = 0. \quad (2.12)$$

In light of (2.11), (2.12) becomes

$$-\frac{\partial E}{\partial v} H'(v) v_x + \frac{\partial E}{\partial v} g'(m) m_x + \frac{\partial E}{\partial m} H'(v) m_x + \frac{\partial E}{\partial m} m H''(v) v_x + \frac{\partial Q}{\partial v} v_x + \frac{\partial Q}{\partial m} m_x = 0. \quad (2.13)$$

Degree	$E(v, m)$
3	$v^3 - 3m^2v$
4	$-2m^2v^2 - \frac{1}{3}m^4 + v^4$
4	$m^3v$
5	$-2m^2v^3 - 3m^4v + v^5$
6	$\frac{45}{7}m^4v^2 - \frac{15}{7}m^2v^4 + \frac{3m^6}{7} + v^6$

TABLE 3. Conservation laws for the forward-forward MFG with  $H(v) = \frac{v^2}{2}$  and  $g(m) = \frac{m^2}{2}$  up to degree 6.

Degree	$E(v, m)$
3	$3m^2v + v^3$
4	$2m^2v^2 - \frac{1}{3}m^4 + v^4$
4	$m^3v$
5	$2m^2v^3 - 3m^4v + v^5$
6	$\frac{45}{7}m^4v^2 + \frac{15}{7}m^2v^4 - \frac{3}{7}m^6 + v^6$

TABLE 4. Conservation laws for the forward-forward MFG with  $H(v) = \frac{v^2}{2}$  and  $g(m) = -\frac{m^2}{2}$  up to degree 6.

Thus,

$$\frac{\partial Q}{\partial v} = \frac{\partial E}{\partial v} H'(v) - \frac{\partial E}{\partial m} m H''(v) \quad \text{and} \quad \frac{\partial Q}{\partial m} = -\frac{\partial E}{\partial v} g'(m) - \frac{\partial E}{\partial m} H'(v). \quad (2.14)$$

Consequently,

$$\frac{\partial}{\partial m} \left( \frac{\partial E}{\partial v} H'(v) - \frac{\partial E}{\partial m} m H''(v) \right) = \frac{\partial}{\partial v} \left( -\frac{\partial E}{\partial v} g'(m) - \frac{\partial E}{\partial m} H'(v) \right). \quad (2.15)$$

This last equation simplifies to

$$\frac{1}{H''(v)} \frac{\partial^2 E}{\partial v^2} + \frac{2H'(v)}{H''(v)g'(m)} \frac{\partial^2 E}{\partial v \partial m} - \frac{m}{g'(m)} \frac{\partial^2 E}{\partial m^2} = 0. \quad (2.16)$$

The preceding equation has a trivial family of solutions,

$$E(v, m) = \alpha v + \beta m, \quad \alpha, \beta \in \mathbb{R}.$$

Moreover, (2.16) admits a solution of the form:

$$E(v, m) = H(v) + P(m)$$

with  $P(m)$  as in (2.10). In contrast with the backward-forward case, here, if  $g$  is increasing, the previous entropy is convex. This observation is crucial for our proof of convergence of the forward-forward mean-field games with viscosity. For illustration, we consider the case  $H(v) = \frac{v^2}{2}$ . In Tables 3 and 4, we present some polynomial conservation laws for, respectively, a monotone,  $g(m) = \frac{m^2}{2}$ , and an anti-monotone,  $g(m) = -\frac{m^2}{2}$ , quadratic forward-forward MFG. These conservation laws satisfy

$$\frac{\partial^2 E}{\partial v^2} \pm \frac{2v}{m} \frac{\partial^2 E}{\partial v \partial m} \mp \frac{\partial^2 E}{\partial m^2} = 0,$$

where the  $-$  sign corresponds to the monotone case and the  $+$  sign to the anti-monotone case.

## 3. WAVE-TYPE EQUATIONS

Here, we introduce a class of wave-type equations that are equivalent to forward-forward MFGs. Using these equations, we rewrite the forward-forward MFG as a new system of conservation laws. For  $g(m) = m^\alpha$ , this new system depends polynomially in  $\alpha$  in contrast with (2.11) where the dependence on  $\alpha$  is exponential. This new formulation is of interest for the numerical simulation of forward-forward MFGs with a large value  $\alpha$  and substantially simplifies the computation of conserved quantities. Subsequently, we consider the logarithmic nonlinearity and, using a result from DiPerna, we prove the existence of a global solution for the forward-forward problem. Moreover, this solution is bounded in  $L^\infty$ . Finally, also for the logarithmic nonlinearity, we investigate the connection between this new formulation and a class of equations introduced in [4]. In particular, we provide a representation formula for some solutions of the forward-forward MFG and establish the existence of shocks.

**3.1. Wave equations and forward-forward MFGs.** We continue our study of forward-forward MFGs by reformulating (1.3) as a scalar nonlinear wave equation. Here, we assume that  $H, g$  are smooth and  $g$  is either strictly increasing or decreasing; that is,  $g' \neq 0$ . From the first equation in (1.3), we have that

$$m = g^{-1}(u_t + H(u_x)). \quad (3.1)$$

We differentiate (3.1) with respect to  $t$  and  $x$  to obtain, respectively,

$$m_t = (g^{-1})'(u_t + H(u_x))(u_{tt} + H'(u_x)u_{xt}) \quad (3.2)$$

and

$$\begin{aligned} (mH'(u_x))_x &= (g^{-1})'(u_t + H(u_x))(u_{tx} + H'(u_x)u_{xx})H'(u_x) \\ &\quad + g^{-1}(u_t + H(u_x))H''(u_x)u_{xx}. \end{aligned} \quad (3.3)$$

Next, we combine (3.2) and (3.3) and get

$$\begin{aligned} m_t - (mH'(u_x))_x &= (g^{-1})'(u_t + H(u_x))(u_{tt} + H'(u_x)u_{xt}) \\ &\quad - (g^{-1})'(u_t + H(u_x))(u_{tx} + H'(u_x)u_{xx})H'(u_x) \\ &\quad - g^{-1}(u_t + H(u_x))H''(u_x)u_{xx}. \end{aligned}$$

Hence, the second equation in (2.11) yields

$$(g^{-1})'(u_t + H(u_x))(u_{tt} - (H'(u_x))^2 u_{xx}) = g^{-1}(u_t + H(u_x))H''(u_x)u_{xx},$$

or, equivalently,

$$u_{tt} = ((H'(u_x))^2 + g'(g^{-1}(u_t + H(u_x))))g^{-1}(u_t + H(u_x))H''(u_x)u_{xx}; \quad (3.4)$$

that is,

$$u_{tt} = ((H'(u_x))^2 + mg'(m)H''(u_x))u_{xx}. \quad (3.5)$$

Thus, (2.11) is equivalent to the nonlinear second-order equation (3.5) coupled with (3.1). Moreover, if  $g$  is increasing, the preceding equation is hyperbolic. In the particular case where  $g(m) = \ln m$ , (3.5) takes the simpler form

$$u_{tt} = ((H'(u_x))^2 + H''(u_x))u_{xx}. \quad (3.6)$$

**3.2. A new system of conservation laws.** Now, we consider the wave equations introduced in the preceding section and reformulate them as a new system of conservation laws. For that, we set  $v = u_x$  and  $w = u_t$ . Then, (3.5) is equivalent to

$$\begin{cases} v_t = w_x, \\ w_t = ((H'(v))^2 + g'(g^{-1}(w + H(v)))g^{-1}(w + H(v))H''(v)) v_x. \end{cases} \quad (3.7)$$

We set

$$\phi(v, w) = (H'(v))^2 + g'(g^{-1}(w + H(v)))g^{-1}(w + H(v))H''(v).$$

Accordingly, (3.7) becomes

$$\begin{cases} v_t = w_x, \\ w_t = \phi(v, w)v_x. \end{cases} \quad (3.8)$$

In the sequel, we choose

$$H(v) = \frac{v^2}{2} \quad \text{and} \quad g(m) = m^\alpha. \quad (3.9)$$

Consequently, we have that

$$mg'(m) = \alpha g(m).$$

Therefore, (3.8) takes the form

$$\begin{cases} v_t = w_x, \\ w_t = (v^2 + \alpha(w + v^2)) v_x. \end{cases}$$

Next, we search for a conserved quantity,  $F(v, w)$ , for the preceding system. Arguing as before, we see that  $F$  is conserved if and only if

$$\frac{\partial^2 F}{\partial v^2} = \frac{\partial}{\partial w} \left( \frac{\partial F}{\partial w} \phi(v, w) \right), \quad (3.10)$$

where  $\phi(v, w) = v^2 + \alpha(w + v^2)$ . A particular solution of (3.10) is

$$F(v, w) = w + \frac{\alpha}{2}v^2. \quad (3.11)$$

Accordingly, we set

$$z(x, t) = w(x, t) + \frac{\alpha}{2}v^2(x, t). \quad (3.12)$$

Thus, we have that

$$\begin{aligned} z_t &= w_t + \alpha v v_t \\ &= \left( v^2 + \alpha \left( w + \frac{v^2}{2} \right) \right) v_x + \alpha v w_x \\ &= \left( 1 + \frac{\alpha}{2} \right) v^2 v_x + \alpha v_x w + \alpha v w_x \\ &= \left( \frac{1}{3} + \frac{\alpha}{6} \right) v_x^3 + \alpha (v w)_x \\ &= \left( \left( \frac{1}{3} + \frac{\alpha}{6} \right) v^3 + \alpha v w \right)_x. \end{aligned}$$

Hence, we obtain the following equivalent system of conservation laws

$$\begin{cases} z_t = \left( \left( \frac{1}{3} + \frac{\alpha}{6} - \frac{\alpha^2}{2} \right) v^3 + \alpha v z \right)_x, \\ v_t = \left( z + \frac{\alpha}{2}v^2 \right)_x. \end{cases} \quad (3.13)$$



Degree	$E(z, v)$
2	$vz$
4	$3\alpha^2 v^4 - \alpha v^4 - 12\alpha v^2 z - 2v^4 - 12z^2$
5	$v(9\alpha^2 v^4 - 3\alpha v^4 - 20\alpha v^2 z - 6v^4 - 60z^2)$
6	$6\alpha^3 v^6 - 2\alpha^2 v^6 - 4\alpha v^6 + 5\alpha^2 v^4 z - 5\alpha v^4 z - 60\alpha v^2 z^2 - 10v^4 z - 20z^3$

TABLE 5. Conservation laws for the modified forward-forward MFG (3.13) up to degree 6.

We observe that  $\alpha$  is no longer in the exponent of the foregoing equation. Therefore, the growth of the nonlinearity becomes polynomial with a fixed degree for any exponent  $\alpha$ . This property is relevant for the numerical analysis and simulation of these games. Moreover, in this formulation, we obtain further polynomial conservation laws for (3.13) shown in Table 5.

**3.3. Forward-forward MFGs with a logarithmic nonlinearity – existence of a solution.** Here, we prove the existence of a solution of (3.6) for a quadratic Hamiltonian. For our proof, we use the ideas in the preceding subsection and rewrite (3.6) as a system of conservation laws. The system we consider here is a special case of the ones investigated in [15], in the whole space, and in [14], in the periodic case. More precisely, we examine the system

$$\begin{cases} v_t - w_x = 0 \\ w_t - \sigma(v)_x = 0 \end{cases} \quad (3.14)$$

with the initial conditions

$$\begin{cases} v(x, 0) = v_0(x) \\ w(x, 0) = w_0(x). \end{cases} \quad (3.15)$$

Here,  $\sigma : \mathbb{R} \rightarrow \mathbb{R}$  is a  $C^2$  function,  $\sigma' > 0$ ,  $(v, w)$  is the unknown and  $(x, t) \in \mathbb{T} \times [0, T]$ . We consider initial data  $v_0, w_0 \in L^\infty(\mathbb{T})$ . As pointed out in [14], if (3.14) has a  $C^1$  solution then there exists  $u$  such that  $w = u_t$ ,  $v = u_x$  and a straightforward computation yields

$$u_{tt} - (\sigma(u_x))_x = 0. \quad (3.16)$$

In addition, for a quadratic Hamiltonian,  $H(p) = \frac{p^2}{2}$ , (3.16) is equivalent to (3.6) for

$$\sigma(z) = z + \frac{z^3}{3}. \quad (3.17)$$

By proving the existence of a solution to (3.6), we get a solution of the corresponding forward-forward MFG.

In [15], the author considers the viscosity approximation

$$\begin{cases} v_t^\varepsilon - w_x^\varepsilon = \varepsilon v_{xx} \\ v_t^\varepsilon - \sigma(u^\varepsilon)_x = \varepsilon w_{xx} \end{cases} \quad (3.18)$$

and proves that, in the limit  $\varepsilon \rightarrow 0$ ,  $(u^\varepsilon, v^\varepsilon)$  converges to a solution of (3.14). For the reader convenience, we reproduce a result from [14] that ensures the existence of a solution of (3.14) in  $\mathbb{T} \times [0, T]$ .

**Theorem 3.1.** *Let  $\sigma$  be given by (3.17). Suppose that  $v_0, w_0 \in L^\infty(\mathbb{T})$ . Then (3.14) has a weak solution  $v, w \in L^\infty(\mathbb{T} \times [0, T])$ .*

*Proof.* The theorem follows from the results in [14] because  $\sigma' > 0$  and  $\sigma''$  vanishes at a single point. Furthermore, as shown in [15], because

$$z\sigma''(z) > 0 \quad \forall z \neq 0 \quad (3.19)$$

and the initial data belongs to  $L^\infty(\mathbb{T} \times [0, T])$ , the theory of invariant regions developed in [11] ensures that

$$\|v\|_{L^\infty(\mathbb{R} \times [0, T])} + \|w\|_{L^\infty(\mathbb{R} \times [0, T])} \leq C.$$

□

**3.4. Logarithmic forward-forward MFGs and Hamilton-Jacobi flows.** We end this section with a brief discussion of the connection between the logarithmic forward-forward MFG and a class of Hamilton-Jacobi flows introduced in [4]. As in discussed in that reference, we consider the Hamilton-Jacobi equation

$$u_t + G(u_x) = 0. \quad (3.20)$$

Assuming smoothness in the equation, we differentiate respectively with respect to  $x$  and  $t$  to obtain:

$$u_{tx} + G'(u_x)u_{xx} = 0 \quad (3.21)$$

and

$$u_{tt} + G'(u_x)u_{tx} = 0. \quad (3.22)$$

Next, we combine (3.21) and (3.22) to get

$$u_{tt} - [G'(u_x)]^2 u_{xx} = 0. \quad (3.23)$$

Finally, we set

$$G(p) = \begin{cases} \frac{1}{2}[p\sqrt{1+p^2} + \operatorname{arcsinh}(p)] & p \geq 0 \\ -\frac{1}{2}[p\sqrt{1+p^2} + \operatorname{arcsinh}(p)] & p < 0, \end{cases} \quad (3.24)$$

so that (3.23) becomes

$$u_{tt} - (1 + u_x^2)u_{xx} = 0.$$

We observe that  $G$  is convex. Thus, we can compute the solution of (3.20) by using the Lax-Hopf formula. For that, we introduce the Legendre transform

$$G^*(v) = \sup_p pv - G(p)$$

and, according to the Lax-Hopf formula, we get the following representation for the solution of (3.20)

$$u(x, t) = \inf_y tG^*\left(\frac{x-y}{t}\right) + u(y, 0). \quad (3.25)$$

If  $u(x, 0)$  is differentiable, so is  $u(x, t)$  for  $0 < t < T^*$ , where  $T^*$  is the time of the first shock.

Now, we set

$$m = e^{H(u_x(x, t)) - G(u_x(x, t))}.$$

Then, for smooth enough solutions, a simple calculation gives

$$m_t - (mu_x)_x = 0.$$

Thus, we see that  $u$  and  $m$  solve the forward-forward MFG

$$\begin{cases} u_t + \frac{u_x^2}{2} = \ln m \\ m_t - (mu_x)_x = 0. \end{cases} \quad (3.26)$$

Finally, because (3.23) depends only on the  $G'(u_x)^2$ , we can repeat the discussion above for the equation

$$u_t - G(u_x) = 0,$$

and obtain another explicit solution.

The examples we discuss in this section show that (3.26) develops shocks in finite time as the regularity of  $u$  is at best the regularity of the solutions of the Hamilton-Jacobi equation (3.20). Moreover, the convergence results for Hamilton-Jacobi equations (see, for example, [6, 13, 16, 26, 30]) show that the function  $u$  given by (3.25) converges (up to additive constants) as  $t \rightarrow \infty$  to a stationary solution of

$$G(u_x) = \overline{G}.$$

#### 4. PARABOLIC MFGs

In Section 2.2, we examined the first-order forward-forward MFGs ( $\varepsilon = 0$ ) and determined several conserved quantities (entropies). In the parabolic ( $\varepsilon > 0$ ) case, these entropies are dissipated. Here, we use this dissipation to establish the long-time convergence of solutions.

As before, by differentiating (1.3) with respect to  $x$ , we get

$$\begin{cases} v_t + (g(m) - H(v))_x = \varepsilon v_{xx}, \\ m_t - (mH'(v))_x = \varepsilon m_{xx}, \end{cases} \quad (4.1)$$

where  $v = u_x$ . We assume that  $g$  is  $C^1$  and strictly increasing, and that  $H$  is  $C^2$  and strictly convex; that is,  $H''(v) > 0$  for all  $v \in \mathbb{R}$ . Additionally, we impose

$$\int_{\mathbb{T}} v(x, 0) dx = 0, \quad \int_{\mathbb{T}} m(x, 0) dx = 1. \quad (4.2)$$

The foregoing conditions are natural because  $v$  is the derivative of a periodic function,  $u$ , and  $m$  is a probability density. A straightforward computation yields the following result.

**Lemma 4.1.** Suppose that  $v, m \in C^2(\mathbb{T} \times (0, +\infty)) \cap C(\mathbb{T} \times (0, +\infty))$  solve (4.1). Furthermore, let  $E(v, m)$  be a  $C^2$  entropy for (2.11); that is,  $E(v, m)$  satisfies (2.16). Then,

$$\frac{d}{dt} \int_{\mathbb{T}} E(v, m) dx = -\varepsilon \int_{\mathbb{T}} (v_x, m_x)^T D^2 E(v, m) (v_x, m_x) dx. \quad (4.3)$$

Now, let  $P(m)$  be as in (2.10). Note that  $P$  is strictly convex when  $g$  is strictly increasing.

**Lemma 4.2.** Let  $\varepsilon > 0$ . Suppose  $v, m \in C^2(\mathbb{T} \times (0, +\infty)) \cap C(\mathbb{T} \times (0, +\infty))$  solve (4.1) and satisfy (4.2). Then, for all  $t \geq 0$ , we have that

$$\int_{\mathbb{T}} v(x, t) dx = 0, \quad \int_{\mathbb{T}} m(x, t) dx = 1. \quad (4.4)$$

Furthermore, if  $g$  is increasing, we have that

$$\frac{d}{dt} \int_{\mathbb{T}} H(v(x, t)) + P(m(x, t)) dx \quad (4.5)$$

$$= -\varepsilon \int_{\mathbb{T}} H''(v(x, t)) v_x^2(x, t) + P''(m(x, t)) m_x^2(x, t) dx \leq 0. \quad (4.6)$$

*Proof.* In Section 2.2, we observed that  $E_0(v, m) = v$ ,  $E_1(v, m) = m$ , and  $E_2(v, m) := H(v) + P(m)$  are entropies for (2.11). Hence, we apply (4.3) to  $E_0, E_1$  and  $E_2$  and obtain (4.4) and (4.5). The inequality in (4.5) follows from the convexity of  $H$  and  $P$ .  $\square$

**4.1. Poincaré-type inequality.** To establish the long-time convergence, we need the following Poincaré-type inequality:

**Theorem 4.3.** *Let  $I \subset \mathbb{R}$  be an open interval and  $\Phi \in C^2(I)$  be a strictly convex function. Furthermore, let  $\Psi \in C^1(I)$  be such that*

$$\Psi'(s) = \sqrt{\Phi''(s)}, \quad s \in I. \quad (4.7)$$

*Then, for every  $f : \mathbb{T} \rightarrow I$ ,  $f \in C^1(\mathbb{T})$ , we have*

$$\int_{\mathbb{T}} \Phi(f(x)) dx - \Phi \left( \int_{\mathbb{T}} f(x) dx \right) \leq C_{\Phi}(a, b) \int_{\mathbb{T}} \Phi''(f(x)) f'(x)^2 dx, \quad (4.8)$$

*where  $a = \min_{\mathbb{T}} f$ ,  $b = \max_{\mathbb{T}} f$ , and*

$$C_{\Phi}(a, b) = \frac{\Phi(a) + \Phi(b) - 2\Phi\left(\frac{a+b}{2}\right)}{(\Psi(b) - \Psi(a))^2}. \quad (4.9)$$

*Moreover, if*

$$C_{\Phi} = \sup_{a, b \in I} C_{\Phi}(a, b) < \infty, \quad (4.10)$$

*then*

$$\int_{\mathbb{T}} \Phi(f(x)) dx - \Phi \left( \int_{\mathbb{T}} f(x) dx \right) \leq C_{\Phi} \int_{\mathbb{T}} \Phi''(f(x)) f'(x)^2 dx \quad (4.11)$$

*for all  $f : \mathbb{T} \rightarrow I$ ,  $f \in C^1(\mathbb{T})$ .*

*Proof.* Because (4.11) is an immediate consequence of (4.8), we only need to prove the latter inequality. For that, next, we show that for every  $f : \mathbb{T} \rightarrow I$ ,  $f \in C^1(\mathbb{T})$ , such that  $a = \min_{\mathbb{T}} f$  and  $b = \max_{\mathbb{T}} f$ , we have

$$\int_{\mathbb{T}} \Phi(f(x)) dx - \Phi \left( \int_{\mathbb{T}} f(x) dx \right) \leq \Phi(a) + \Phi(b) - 2\Phi\left(\frac{a+b}{2}\right) \quad (4.12)$$

and

$$\int_{\mathbb{T}} \Phi''(f(x)) f'(x)^2 dx \geq (\Psi(b) - \Psi(a))^2. \quad (4.13)$$

If  $a = b$ ,  $f$  is constant and the result is trivial. Thus, we assume  $a < b$ . Let  $A = \int_{\mathbb{T}} f(x) dx$ . We have that  $a \leq A \leq b$ . Furthermore, because  $\Phi$  is convex, we have that

$$\Phi(s) \leq \frac{b-s}{b-a} \Phi(a) + \frac{s-a}{b-a} \Phi(b) =: L(s), \quad \forall s \in [a, b].$$

Now, we observe that  $\Phi(s) - L(s)$  is a convex function that vanishes at  $s = a, b$ . Accordingly, for  $a < s < \frac{a+b}{2}$  there exists  $\lambda > \frac{1}{2}$  such that

$$\frac{a+b}{2} = \lambda s + (1-\lambda)b.$$

Therefore,

$$\begin{aligned} \Phi\left(\frac{a+b}{2}\right) - L\left(\frac{a+b}{2}\right) &\leq \lambda(\Phi(s) - L(s)) + (1-\lambda)(\Phi(b) - L(b)) \\ &= \lambda(\Phi(s) - L(s)). \end{aligned} \quad (4.14)$$

Arguing in a similar way for  $\frac{a+b}{2} \leq s < b$ , we see that (4.14) also holds for some  $\lambda > \frac{1}{2}$ . Consequently, we have

$$\begin{aligned}
L(s) - \Phi(s) &\leq 2 \left( L\left(\frac{a+b}{2}\right) - \Phi\left(\frac{a+b}{2}\right) \right) \\
&= \Phi(a) + \Phi(b) - 2\Phi\left(\frac{a+b}{2}\right)
\end{aligned}$$

for all  $s \in [a, b]$ . Hence, we get

$$\int_{\mathbb{T}} \Phi(f(x)) dx \leq \int_{\mathbb{T}} L(f(x)) dx = L\left(\int_{\mathbb{T}} f(x) dx\right) = L(A).$$

Therefore,

$$\int_{\mathbb{T}} \Phi(f(x)) dx - \Phi\left(\int_{\mathbb{T}} f(x) dx\right) \leq L(A) - \Phi(A) \leq \Phi(a) + \Phi(b) - 2\Phi\left(\frac{a+b}{2}\right).$$

Suppose  $f(x_0) = a$  and  $f(x_1) = b$ . Then, we have that

$$\begin{aligned}
\int_{\mathbb{T}} \Phi''(f(x)) f'(x)^2 dx &= \int_{\mathbb{T}} \left( \frac{d\Psi(f(x))}{dx} \right)^2 dx \geq \left( \int_{\mathbb{T}} \left| \frac{d\Psi(f(x))}{dx} \right| dx \right)^2 \\
&\geq \left( \int_{x_0}^{x_1} \left| \frac{d\Psi(f(x))}{dx} \right| dx \right)^2 \geq \left| \int_{x_0}^{x_1} \frac{d\Psi(f(x))}{dx} dx \right|^2 \\
&= (\Psi(b) - \Psi(a))^2.
\end{aligned}$$

□

Next, we present some convex functions  $\Phi$  for which (4.10) holds.

**Proposition 4.4.** Let  $I$  and  $\Phi \in C^2(I)$  be one of the following:

1.  $I = (0, \infty)$ ,  $\Phi(s) = s^p$ , where  $p > 1$ .
2.  $I = (0, \infty)$ ,  $\Phi(s) = s^p$ , where  $p < 0$ .
3.  $I = (0, \infty)$ ,  $\Phi(s) = -s^p$ , where  $0 < p < 1$ .
4.  $I = (0, \infty)$ ,  $\Phi(s) = -\ln s$ .
5.  $I = (0, \infty)$ ,  $\Phi(s) = s \ln s$ .
6.  $I = \mathbb{R}$ ,  $\Phi(s) = s^{2n}$ , where  $n \in \mathbb{N}$ .
7.  $I = \mathbb{R}$ ,  $\Phi(s) = e^{\alpha s}$ , where  $\alpha \in \mathbb{R}$ .

Then,  $C_\Phi$  defined in (4.10) is finite. Consequently, (4.11) holds.

*Proof.* The proof of the preceding result is elementary though tedious, and we omit it here. □

**4.2. Stability of Jensen's inequality.** The proof of the long-time convergence of the solutions of (4.1) is based on the following stability property of Jensen's inequality:

**Theorem 4.5.** Let  $I \subset \mathbb{R}$  be an open interval, not necessarily bounded, and  $\Phi \in C(I)$  a strictly convex function. Furthermore, let  $A \in I$  and  $f_t : \mathbb{T} \rightarrow I$ ,  $\{f_t\}_{t>0} \subset C(\mathbb{T})$ , be such that, for all  $t \geq 0$ ,

$$\int_{\mathbb{T}} f_t(x) dx = A$$

and

$$\lim_{t \rightarrow \infty} \int_{\mathbb{T}} \Phi(f_t(x)) dx - \Phi(A) = 0.$$

Then,

$$\lim_{t \rightarrow \infty} \int_{\mathbb{T}} |f_t(x) - A| dx = 0.$$

**Remark 4.6.** Note that we do not impose uniform  $L^\infty$  bounds on the family  $\{f_t\}_{t>0}$ .

Before proving Theorem 4.5, we need the following technical lemma. We recall that  $\mathcal{L}^1$  denotes the one-dimensional Lebesgue measure.

**Lemma 4.7.** *Let  $I \subset \mathbb{R}$  be some interval and  $\Phi \in C(I)$  a convex function. Then, for every  $f \in C(I)$ , we have that*

$$\int_{\mathbb{T}} \Phi(f(x)) dx - \Phi \left( \int_{\mathbb{T}} f(x) dx \right) \geq p\Phi(A_1) + q\Phi(A_2) - (p+q)\Phi(A) \geq 0, \quad (4.15)$$

where

$$A = \int_{\mathbb{T}} f(x) dx, \quad p = \mathcal{L}^1(\{f < A\}), \quad q = \mathcal{L}^1(\{f \geq A\}), \quad (4.16)$$

$$A_1 = \int_{f < A} f(x) dx = A - \frac{\gamma(f)}{p}, \quad A_2 = \int_{f \geq A} f(x) dx = A + \frac{\gamma(f)}{q}, \quad (4.17)$$

and

$$\gamma(f) = \int_{\mathbb{T}} (f(x) - A)^- dx = \int_{\mathbb{T}} (f(x) - A)^+ dx = \frac{1}{2} \int_{\mathbb{T}} |f(x) - A| dx. \quad (4.18)$$

*Proof.* By rearranging (4.15) and observing that  $p+q=1$ , we get the inequality

$$\begin{aligned} & \int_{f < A} \Phi(f(x)) dx + \int_{f \geq A} \Phi(f(x)) dx \\ & \geq \mathcal{L}^1(\{f(x) < A\}) \Phi \left( \int_{f < A} f(x) dx \right) + \mathcal{L}^1(\{f(x) \geq A\}) \Phi \left( \int_{f \geq A} f(x) dx \right). \end{aligned}$$

The result follows by observing that the preceding inequality is a consequence of Jensen's inequality.  $\square$

Now, we are ready to prove Theorem 4.5.

*Proof of Theorem (4.5).* Let  $p_t, q_t, A_1^t, A_2^t$  and  $\gamma_t := \gamma(f_t)$  be as in (4.16)-(4.18) for  $f = f_t$ . From (4.15), we have that

$$p_t \Phi(A_1^t) + q_t \Phi(A_2^t) - (p_t + q_t) \Phi(A) \rightarrow 0.$$

By contradiction, we assume that  $f_t$  does not converge to the common average value  $A$ . Then, without loss of generality, we can assume that there exists  $\varepsilon_0 > 0$  such that

$$\gamma_{t_n} \geq \varepsilon_0 > 0$$

for some sequence  $t_n \rightarrow \infty$ . Consequently,

$$|A_1^{t_n} - A| = \frac{\gamma_{t_n}}{p_{t_n}} \geq \varepsilon_0$$

and

$$|A_2^{t_n} - A| = \frac{\gamma_{t_n}}{q_{t_n}} \geq \varepsilon_0.$$

Because  $\Phi$  is strictly convex, we have that

$$k = \inf_{|s-A| \geq \varepsilon_0} \frac{\Phi(s) - \Phi(A) - (s-A)\alpha}{|s-A|} = \min_{|s-A|=\varepsilon_0} \frac{\Phi(s) - \Phi(A) - (s-A) \cdot \alpha}{|s-A|} > 0$$

for any  $\alpha$  in the subdifferential  $\partial^- \Phi(A)$ . Therefore,

$$\begin{aligned} p_{t_n} \Phi(A_1^{t_n}) + q_{t_n} \Phi(A_2^{t_n}) - (p_{t_n} + q_{t_n}) \Phi(A) &= p_{t_n} (\Phi(A_1^{t_n}) - \Phi(A) - (A_1^{t_n} - A)\alpha) \\ &\quad + q_{t_n} (\Phi(A_2^{t_n}) - \Phi(A) - (A_2^{t_n} - A)\alpha) \\ &\geq k p_{t_n} |A_1^{t_n} - A| + k q_{t_n} |A_2^{t_n} - A| \\ &= k \gamma_{t_n} \geq k \varepsilon_0, \end{aligned}$$

which is a contradiction.  $\square$

If we have uniform  $L^\infty$  bounds, we have the following stronger stability property for Jensen's inequality:

**Theorem 4.8.** *Let  $I \subset \mathbb{R}$  be some interval and  $\Phi \in C(I)$  a strictly convex function. Furthermore, let  $a < b$  be real numbers and consider a family of functions  $f_t : \mathbb{T} \rightarrow I$ ,  $\{f_t\}_{t>0} \subset C(\mathbb{T})$ , such that*

$$a \leq f_t(x) \leq b, \quad \forall x \in \mathbb{T}, \quad \forall t > 0,$$

and

$$\lim_{t \rightarrow \infty} \int_{\mathbb{T}} \Phi(f_t(x)) dx - \Phi \left( \int_{\mathbb{T}} f_t(x) dx \right) = 0.$$

Then, we have that

$$\lim_{t \rightarrow \infty} \int_{\mathbb{T}} |f_t(x) - A_t| dx = 0, \tag{4.19}$$

where  $A_t = \int_{\mathbb{T}} f_t(x) dx$ . Consequently,

$$\lim_{t \rightarrow \infty} \int_{\mathbb{T}} |f_t(x) - A_t|^p dx = 0 \tag{4.20}$$

for all  $p > 1$ .

*Proof.* Because  $f_t$  is bounded, (4.20) follows from (4.19). Therefore, we only need to prove the latter. Let  $p_t, q_t, A_1^t, A_2^t$  and  $\gamma_t := \gamma(f_t)$  be as in (4.16)-(4.18) for  $f = f_t$ . By contradiction, we assume that there exists  $\varepsilon_0 > 0$  such that

$$\gamma_{t_n} \geq \varepsilon_0 > 0,$$

for some  $t_n \rightarrow \infty$ . Accordingly,

$$|A_1^{t_n} - A_{t_n}| = \frac{\gamma_{t_n}}{p_{t_n}} \geq \varepsilon_0,$$

and

$$|A_2^{t_n} - A_{t_n}| = \frac{\gamma_{t_n}}{q_{t_n}} \geq \varepsilon_0.$$

We have that  $a \leq A_t, A_1^t, A_2^t \leq b$ . Therefore, by compactness, we can assume that

$$A_1^{t_n} \rightarrow A_1, \quad A_2^{t_n} \rightarrow A_2, \quad A_{t_n} \rightarrow A, \quad p_{t_n} \rightarrow p, \quad q_{t_n} \rightarrow q,$$

extracting a subsequence if necessary. Moreover, we have that

$$|A_1 - A|, |A_2 - A| \geq \varepsilon_0 > 0.$$

Furthermore, since  $\Phi$  is continuous, we have that

$$p_{t_n} \Phi(A_1^{t_n}) + q_{t_n} \Phi(A_2^{t_n}) - (p_{t_n} + q_{t_n}) \Phi(A_{t_n}) \rightarrow p \Phi(A_1) + q \Phi(A_2) - (p + q) \Phi(A) = 0,$$

using (4.15). Note that

$$p_t A_1^t + q_t A_2^t = (p_t + q_t) A_t$$

for all  $t > 0$ . Hence,

$$p A_1 + q A_2 = (p + q) A.$$

Next, since  $\Phi$  is strictly convex, we get that  $p = 0$  or  $q = 0$ . But then  $p_{t_n} \rightarrow 0$  or  $q_{t_n} \rightarrow 0$ . Suppose  $p_{t_n} \rightarrow 0$ . Then,

$$\varepsilon_0 \leq \gamma_{t_n} = \int_{f_{t_n} < A_{t_n}} |f_{t_n}(x) - A_{t_n}| dx \leq (b - a) \mathcal{L}^1(\{f_{t_n} < A_{t_n}\}) = (b - a) p_{t_n},$$

which is a contradiction. Similarly, we get a contradiction if  $q_{t_n} \rightarrow 0$ .  $\square$

**4.3. Parabolic forward-forward MFGs – convergence.** Now, we are ready to prove the convergence result for (4.1).

**Theorem 4.9.** *Let  $H \in C^2(\mathbb{R})$  be strictly convex and  $g \in C^1((0, \infty))$  be strictly increasing. Suppose that  $C_H, C_P < \infty$  (see (4.10)), where  $P$  is as in (2.10). Furthermore, let  $v, m \in C^2(\mathbb{T} \times (0, +\infty)) \cap C(\mathbb{T} \times [0, +\infty))$ ,  $m > 0$ , solve (4.1) and satisfy (4.2). Then, we have that*

$$\lim_{t \rightarrow \infty} \int_{\mathbb{T}} |v(x, t)| dx = 0, \quad \lim_{t \rightarrow \infty} \int_{\mathbb{T}} |m(x, t) - 1| dx = 0. \quad (4.21)$$

Moreover, if

$$\sup_{t \geq 0} \|v(\cdot, t)\|_{C(\mathbb{T})} \quad \text{and} \quad \sup_{t \geq 0} \|m(\cdot, t)\|_{C(\mathbb{T})} < \infty,$$

then, for all  $1 < p < \infty$ ,

$$\lim_{t \rightarrow \infty} \int_{\mathbb{T}} |v(x, t)|^p dx = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \int_{\mathbb{T}} |m(x, t) - 1|^p dx = 0. \quad (4.22)$$

*Proof.* Let  $C_0 := \max\{C_H, C_P\}$ . Let

$$I(t) = \int_{\mathbb{T}} H(v(x, t)) + P(m(x, t)) dx - H(0) - P(1).$$

From (4.4), (4.5), and (4.11), we have that

$$\begin{aligned} \frac{dI(t)}{dt} &= -\varepsilon \int_{\mathbb{T}} H''(v(x, t)) v_x^2(x, t) + P''(m(x, t)) m_x^2(x, t) dx \\ &\leq -\frac{\varepsilon}{C_0} \left( \int_{\mathbb{T}} H(v(x, t)) dx - H(0) + \int_{\mathbb{T}} P(m(x, t)) dx - P(1) \right) \\ &= -\frac{\varepsilon}{C_0} I(t). \end{aligned}$$



Therefore, we get

$$I(t) \leq e^{-\frac{\varepsilon}{c_0}t} I(0) \quad \forall t \geq 0,$$

which yields

$$\lim_{t \rightarrow \infty} I(t) = 0.$$

Furthermore, by Jensen's inequality, we have that

$$\int_{\mathbb{T}} H(v(x, t)) dx - H(0) \leq I(t),$$

and

$$\int_{\mathbb{T}} P(m(x, t)) dx - P(1) \leq I(t).$$

Therefore, we get

$$\lim_{t \rightarrow \infty} \int_{\mathbb{T}} H(v(x, t)) dx - H(0) = \lim_{t \rightarrow \infty} \int_{\mathbb{T}} P(m(x, t)) dx - P(1) = 0,$$

and we conclude using Theorem 4.5.  $\square$

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